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Parametrizations of the Dark Energy Density and Scalar Potentials

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We develop a theoretical method of constructing the scalar (quintessence or phantom) potential directly from the dimensionless dark energy function $X(z)$, the dark energy density in units of its present value. We apply our method to two parametrizations of the dark energy density, the quiescence-Lambda ansatz and the generalized Chaplygin gas model, and discuss some features of the constructed potentials.

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Recent observations of type Ia supernovae suggest that the expansion of the universe is accelerating and that two-thirds of the total energy density exists in a dark energy component with negative pressure.^{1,2} In addition, measurements of the cosmic microwave background³ and the galaxy power spectrum⁴ also indicate the existence of the dark energy. The simplest candidate for the dark energy is a cosmological constant Λ , which has pressure $P_\Lambda = -\rho_\Lambda$. Specifically, a reliable model should explain why the present amount of the dark energy is so small compared with the fundamental scale (fine-tuning problem) and why it is comparable with the critical density today (coincidence problem). The cosmological constant suffers from both these problems. One possible approach to constructing a viable model for dark energy is to associate it with a slowly evolving and spatially homogeneous scalar field ϕ , called “quintessence”.^{5,6,7,8,9} Such a model for a broad class of potentials can give the energy density converging to its present value for a wide set of initial conditions in the past and possess tracker behavior (see, e.g., Ref. 10, 11, 12, 13 for reviews with more complete lists of references).

The dark energy is characterized by its equation of state w and its energy density ρ , which are in general functions of redshift z in scalar field models. For a minimally coupled scalar field model, the quintessence potential $V(\phi)$ may be reconstructed from supernova observations^{10,14,15,16,17,18} and directly from the effective equation of state function $w(z)$.¹⁹ Given an evolution of the universe, the potential for a scalar and tachyon field can be constructed.²⁰ Moreover, the cosmological evolutions may be drastically different for arbitrary initial conditions on

the tachyon field.²⁰ Recently, the reconstruction method has been studied in the k-essence models,^{21,22,23} tachyon models,^{24,21} 5D cosmological models^{25,26,27} and scalar-tensor theories.^{28,29}

Although the equation of state $w(z)$ has been generally chosen to parametrize the dark energy, there are several reasons why the parametrization by the energy density $\rho(z)$, which depends on its equation of state $w(z)$ through an integral, gives a better one. Firstly, it can be constrained more tightly than $w(z)$ for given observational data.^{30,31} Secondly, parametrization of the energy density provides a more flexible approach, which can determine the properties of dark energy in a model independent manner. Finally it was shown that assuming that w is constant or greater than -1 can lead to gross errors in estimating the true equation of state.³² This smearing effect can be decreased by using the dark energy density $\rho(z)$. In this letter we develop a theoretical method of constructing the scalar potential $V(\phi)$ directly from the dark energy density $\rho(z)$, which connects the energy density parametrization to the physically effective field theory. We apply this method to two parametrizations and discuss some features of the resulting potentials.

Our method is new in that it relates directly the scalar potential to the dark energy function and it is also extended to the phantom field models. This enables us to construct easily the potential without assuming its form. For instance, in the reconstruction method proposed in Refs. 10, 14, 15, 16, 17, 18, the reconstruction equations relate the potential and the equation of state to measurements of the luminosity distance. The potential may thus be reconstructed by way of the luminosity distance from supernova data. Usually this can be done by assuming the form of a potential. Compared to the methods discussed in Refs. 20 and 19, it is extended to the construction of a phantom field potential. A similar reconstruction of the potential was also considered in Ref. 28, 29 using a single non-canonical scalar field and the Hubble parameter parametrized by time. We generalize this approach by including the contribution of the cold dark matter as well. Moreover our method itself has the advantage of using parametrization of Hubble parameter by redshift which is directly observed, but that in terms of time is not an observed quantity.

We consider a spatially flat FRW universe which is dominated by the non-relativistic matter and a spatially homogeneous scalar field ϕ . The Einstein equation can be written as

$$H^2 = \frac{1}{3M_{pl}^2}(\rho_m + \rho_\phi), \quad (1)$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_{pl}^2}(\rho_m + \rho_\phi + 3P_\phi), \quad (2)$$

where $M_{pl} \equiv (8\pi G)^{-1/2}$ is the reduced Planck mass, a is the scale factor, $H = \dot{a}/a$ is the Hubble parameter, and ρ_m is the matter density. The energy density ρ_ϕ and

pressure P_ϕ of the evolving scalar field ϕ are given by

$$\rho_\phi = \pm \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad (3)$$

$$P_\phi = \pm \frac{1}{2} \dot{\phi}^2 - V(\phi), \quad (4)$$

respectively, where $V(\phi)$ is the scalar field potential. The upper (lower) sign corresponds to a quintessence (phantom) field in Eqs. (3) and (4) and in what follows. Using the Einstein equations and the expressions for ρ_ϕ and P_ϕ , one can obtain

$$V = 3M_{pl}^2 H^2 + M_{pl}^2 \dot{H} - \frac{1}{2} \rho_m, \quad (5)$$

$$\dot{\phi}^2 = \mp \left(2M_{pl}^2 \dot{H} + \rho_m \right), \quad (6)$$

which can be rewritten as

$$V(z) = 3M_{pl}^2 H^2 - M_{pl}^2 (1+z) H \frac{dH}{dz} - \frac{1}{2} \rho_{m0} (1+z)^3, \quad (7)$$

$$\pm \left(\frac{d\phi}{dz} \right)^2 = 2M_{pl}^2 (1+z)^{-1} H^{-1} \frac{dH}{dz} - \rho_{m0} (1+z) H^{-2}, \quad (8)$$

in terms of the redshift $z = -1 + a_0/a$, and we have used the relation $\rho_m = \rho_{m0} (1+z)^3$. Throughout this paper quantities with subscript 0 denote the values at the redshift $z = 0$ (present).

We define dimensionless quantities

$$\tilde{V} \equiv V/\rho_0, \quad \tilde{\phi} \equiv \phi/M_{pl}, \quad (9)$$

where $\rho_0 = \rho_{\phi 0} + \rho_{m0}$ is the total energy density at present time. The construction equations (7) and (8) can then be written as

$$\tilde{V}(z) = (1 - \Omega_{m0}) \left[X(z) - \frac{1}{6} (1+z) \frac{dX(z)}{dz} \right], \quad (10)$$

$$\left(\frac{d\tilde{\phi}}{dz} \right)^2 = \pm (1 - \Omega_{m0}) \frac{1}{(1+z)E^2(z)} \frac{dX(z)}{dz}, \quad (11)$$

where $X(z) \equiv \rho_\phi(z)/\rho_{\phi 0}$ is the dimensionless dark energy function, the dark energy density in units of its present value, $\Omega_{m0} \equiv \rho_{m0}/\rho_0$ is the present day density parameter of matter, and

$$E(z) \equiv [\Omega_{m0} (1+z)^3 + (1 - \Omega_{m0}) X(z)]^{1/2} \quad (12)$$

is the cosmic expansion rate relative to its present value. Given the dark energy function $X(z)$, the construction equations (10) and (11) will allow us to construct the scalar potential $V(\phi)$, quintessence potential for $dX/dz > 0$ and phantom potential for $dX/dz < 0$. The sign of $d\phi/dz$ in fact is arbitrary, as it can be changed by the field redefinition, $\phi \rightarrow -\phi$. So we choose $d\phi/dz < 0$ in the following discussions, that is, the scalar field ϕ increases as the universe expands.

As examples, let us now consider the following two parametrizations: the quiescence- Λ ansatz and the generalized Chaplygin gas model.

(i) Quiescence- Λ ansatz:

$$X(z) = A + (1 - A)(1 + z)^{3(1+w)}, \quad (13)$$

where A and w are constant. This ansatz involves the combination of cosmological constant with “quiescence” (dark energy with constant equation of state parameter), called quiescence- Λ ansatz.^{33,34} Both early on and in the distant future, the dark energy approaches either a constant equation of state w or a constant density, depending on the sign of $(1 + w)$. When the universe is dominated by dark energy, we obtain

$$\tilde{V}(\tilde{\phi}) = A(1 - \Omega_{m0}) \left[1 + \frac{1 - w}{2} \sinh^2 \frac{\sqrt{\pm 3(1 + w)}(\tilde{\phi} - \tilde{\phi}_0)}{2} \right]. \quad (14)$$

For $w < -1$, this result is consistent with that in Ref. 20. Note that $d\tilde{V}/d\tilde{\phi} < 0$ for $w > -1$ and $d\tilde{V}/d\tilde{\phi} > 0$ for $w < -1$. This result implies that the quintessence field rolls down its potential while the phantom field climbs up its potential.

(ii) Generalized Chaplygin gas model:

$$X(z) = \left[A_s + (1 - A_s)(1 + z)^{3(1+\alpha)} \right]^{\frac{1}{1+\alpha}}, \quad (15)$$

where A_s and α are constant. This ansatz can be obtained by modeling the dark energy as generalized Chaplygin gas with an equation of state $P_c = -A/\rho_c$.^{35,36,37} By choosing different ranges for the parameters (A_s, α) , this ansatz can behave as standard dark energy model with asymptotic de-Sitter phase, early phantom model in which $w \ll -1$ at early times and asymptotically approaches $w = -1$ at late times, late phantom model with $w \approx -1$ at early times and $w \rightarrow -\infty$ at late times, and transient model where the present acceleration of the universe is only temporary as it again enters the dust-dominated decelerating phase in future.³⁸ When the universe is dominated by the quintessence-like Chaplygin gas with $A_s < 1$, one gets

$$\begin{aligned} \tilde{V}(\tilde{\phi}) = \frac{1}{2} A_s^{\frac{1}{1+\alpha}} (1 - \Omega_{m0}) & \left[\cosh^{\frac{2}{1+\alpha}} \frac{\sqrt{3}(1 + \alpha)(\tilde{\phi} - \tilde{\phi}_0)}{2} \right. \\ & \left. + \cosh^{\frac{-2\alpha}{1+\alpha}} \frac{\sqrt{3}(1 + \alpha)(\tilde{\phi} - \tilde{\phi}_0)}{2} \right]. \end{aligned} \quad (16)$$

The potential (16) with $\alpha = 1$ corresponds to the original Chaplygin gas model, which was reconstructed in Ref. 35. The generalized model with $\alpha = 0$ give the same reconstructed potential as the quiescence- Λ model with $w = 0$. For the phantom-like

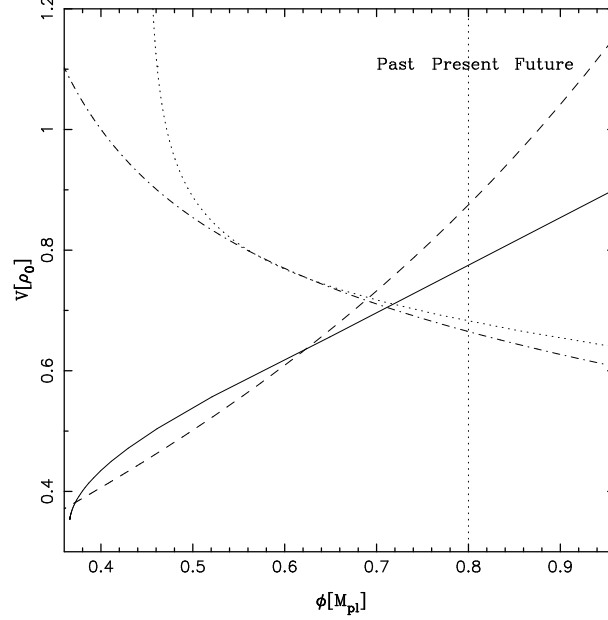


Fig. 1. Constructed scalar potentials for the quiescence- Λ ansatz with $\Omega_{m0} = 0.3$ and $A = 0.5$. The solid, dashed, dotted and dot-dashed lines correspond to $w = -1.4$, $w = -2.0$, $w = -0.9$ and $w = -0.8$ respectively.

Chaplygin gas with $A_s > 1$, we obtain

$$\begin{aligned} \tilde{V}(\tilde{\phi}) = \frac{1}{2} A_s^{\frac{1}{1+\alpha}} (1 - \Omega_{m0}) & \left[\sin^{\frac{2}{1+\alpha}} \frac{\sqrt{3}(1+\alpha)(\tilde{\phi} - \tilde{\phi}_0)}{2} \right. \\ & \left. + \sin^{\frac{-2\alpha}{1+\alpha}} \frac{\sqrt{3}(1+\alpha)(\tilde{\phi} - \tilde{\phi}_0)}{2} \right]. \end{aligned} \quad (17)$$

In Fig. 1, we have plotted the constructed quintessence or phantom potential for the quiescence- Λ ansatz with $\Omega_{m0} = 0.3$ and $A = 0.5$. Clearly, Eq. (13) indicates that the effective equation of state approaches -1 if $w > -1$ and w if $w < -1$. As shown in Fig. 1, in the case of $w > -1$ the constructed quintessence potentials are in the form of a runaway type, in agreement with our previous analysis.¹⁹ On the other hand, for $w < -1$ the potentials increase as the universe expands. The asymptotic form is obtained as follows. The phantom energy density increases with time, so the matter component in Eq. (12) and the first term in Eq. (13) become negligible when $z \rightarrow -1$. We find that the potentials tend to the exponential form:

$$\tilde{V}(\tilde{\phi}) = \frac{1}{2} (1 - \Omega_{m0})(1 - A)(1 - w) \exp \left[\sqrt{-3(1+w)}(\tilde{\phi} - \tilde{\phi}_0) \right], \quad (18)$$

which leads to unwanted future singularity called “Big Rip”. Therefore, in this parametrization (13), the universes either accelerates forever (if $w > -1$) or reaches

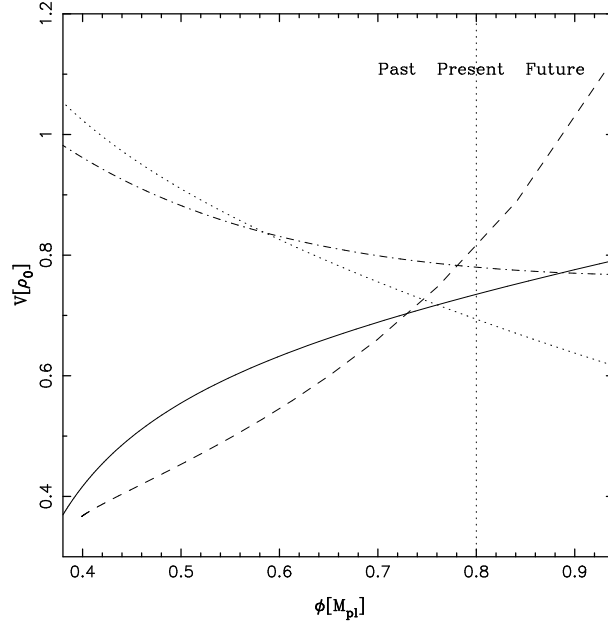


Fig. 2. Constructed scalar potentials for the generalized Chaplygin gas model. The solid, dashed, dotted and dot-dashed lines represent the early phantom model ($\Omega_{m0} = 0.3$, $A_s = 1.1$, $\alpha = -0.8$), late phantom model ($\Omega_{m0} = 0.35$, $A_s = 1.5$, $\alpha = -1.7$), transient model ($\Omega_{m0} = 0.25$, $A_s = 0.85$, $\alpha = -1.2$) and standard model ($\Omega_{m0} = 0.2$, $A_s = 0.95$, $\alpha = 0.2$) respectively.

a Big Rip (if $w < -1$) at late times. In the former case, the quiescence-dominated phase is transient and de-Sitter is a late-time attractor. In the latter case, the phantom dominates the universe in the late times. Thus a potential with a local maximum can not be constructed, in which case the phantom dark energy is transient and de-Sitter is a late-time attractor.^{39,40}

Fig. 2 shows the constructed scalar potentials for the generalized Chaplygin gas model with $\Omega_{m0} = 0.3$. One can see that the potentials tend to be flat for the standard and early phantom models while the potentials become steep as the universe expands for the transient and late phantom models.

In the evaluation of these equations, we have also chosen the initial values of the scalar field $\tilde{\phi}_0 = 0.8$ at the redshift $z = 0$ (present). The value of $\tilde{\phi}_0$ is chosen for the purpose of definiteness. If we shift its value, it simply results in the shift of the value of the scalar field; the potential in Figs. 1 and 2 is shifted horizontally. It has no influence on the evolution of the universe and the shape of the quintessence potential.

Compared to the method proposed in Ref. 19, it is extended to the phantom field with a negative kinetic term. The dark energy density function, $X(z)$, is related

to $w(z)$ as follows:

$$X(z) = \exp \left[3 \int_0^z (1+w) d \ln(1+z) \right], \quad (19)$$

so that $w = -1 + \frac{1}{3}(1+z) d \ln X / dz$. One can see that it is easier to extract $X(z)$ from the data than to extract $w(z)$. Wang and Garavich argued that $X(z)$ should be preferred since it suffers less from the smearing effect that makes constraining $w(z)$ extremely difficult.^{30,31} By precision measure of $X(z)$, we can obtain the scalar potential using the construction method.

In conclusion, we have developed a method of constructing the quintessence and phantom potentials directly from the dark energy density function $X(z)$. From the theoretical viewpoint, the method connects the energy density parametrization to the effective field theory and throws light on the physical nature of dark energy. Then we have considered two parametrizations and constructed the scalar potential.

It is emphasized that the method is invalid when the effective equation of state crosses the phantom divide $w = -1$ since neither a single quintessence field nor a single phantom field realizes the crossing of the phantom divide.⁴¹ But in the quintom model with a quintessence field and a phantom field this case can be realized easily.^{42,43}

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